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LETTER TO THE EDITOR

How Smooth Is the Smoothest Function in a Given Refinable Space?

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A closed subspace V of $L_2 := L_2(\mathbb{R}^d)$ is called *PSI* (principal shift-invariant) if it is the smallest space that contains all the *shifts* (i.e., integer translates) of some function $\phi \in L_2$. Ideally, each function f in such PSI V can be written uniquely as a convergent series

$$f = \sum_{\alpha \in \mathbb{Z}^d} c(\alpha)\phi(\cdot - \alpha)$$

with $\|c\|_{l_2} \sim \|f\|_{L_2}$. In this case one says that the shifts of ϕ form a *Riesz basis* or that they are L_2*-stable*; this is, in particular, the case when these shifts form an orthonormal set.

We are interested here in PSI spaces which are *refinable* in the sense that, for some integer $N > 1$, the space

$$V_{-1} := V(\cdot/N) := \{f(\cdot/N) : f \in V\}$$

is a subspace of V. The role of refinable PSI spaces in the construction of wavelets from multiresolution analysis, as well as in the study of subdivision algorithms is well-known, well-understood, and well-documented (cf. e.g., [6, 4]). The two properties of a refinable PSI space that we compare here are:

(s) the smoothness of the "smoothest" nonzero function $g \in V$.

(ao) the approximation orders provided by V.

This latter notion refers to the decay of the error when approximating smooth functions from dilations of V; roughly speaking, V *provides approximation order* k if

$$\operatorname{dist}(f, V_j) = O(N^{-jk})$$

for every sufficiently smooth function f. Here, $V_j := V(N^j\cdot)$.

One of the early discoveries in this area was the nontrivial observation that for a refinable PSI space V, (s) and (ao) are connected. For example, a result in [7] shows that if ϕ decays rapidly and its shifts are L_2-stable, then V provides approximation order k as soon as ϕ lies in the Sobolev space W_2^{k-1}. A closely related result appears in [4]. More recently, the following is proved in [8]:

RESULT 1. *Let V be an N-refinable PSI space. Then the following conditions are equivalent*:

(a) *V provides approximation order k.*

(b) *There exists $g \in V_r, r > 0$, such that $|\hat{g}| \geqslant$ const > 0 on some neighborhood of the origin, and such that*

$$\operatorname*{ess\,sup}_{\xi \in C} \left(\sum_{\alpha \in \mathbb{Z}^d \setminus \{0\}} |\hat{g}(\xi + 2\pi N^j \alpha)|^2 \right) = O(N^{-2jk}),$$

with C the cube $[-\pi, \pi]^d$.

The second (and more essential) requirement in property (b) of the above result is satisfied by functions g that are sufficiently smooth. It is then correct to say that, with the exclusion of truly pathological examples, refinable PSI spaces that contain smooth functions must provide good approximation orders. But what about the converse?

Reference [8] considers the converse for PSI spaces that are *totally refinable.* For univariate spaces, this simply means that V is N-refinable for *every* integer N. For such spaces, the following is valid:

RESULT 2. *Assume in Result 1 that V is totally refinable. If V provides approximation order k, then there exists nonzero $g \in V$ such that*

$$|\hat{g}(\omega)| = O(|\omega|^{-k}), \text{ as} |\omega| \to \infty.$$

Thus, *for totally refinable spaces, smoothness and approximation orders go hand-in-hand.* Indeed, the best known cases of such spaces are the univariate splines and the multivariate box splines (cf. [2]), and for these spline spaces the rigid connection between the smoothness of the spline and the underlying approximation order of the spline space is classically known.

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However, most refinable spaces are *not* totally refinable. For example, the spaces generated by the compactly supported scaling functions in [5] are not. This leads naturally to the following question: "Must refinable spaces contain sufficiently smooth functions the moment they provide "good" approximation order"? One may observe that condition (b) of Result 1 (which characterizes the approximation orders of V) falls short of implying any smoothness for *general* L_2-functions g. However, the functions that comprise a refinable space are anything but "*general*"!

We therefore decided to look into the question of finding the smoothest function in a refinable space, and to see whether its smoothness matches the approximation order.

Our conclusion in this note is that the implication (ao) $\Rightarrow$ (s) *does not hold for general refinable spaces.* In fact, the proposition we prove below implies that, if V is a space generated by any of the scaling functions ϕ considered in [5], then the decay rate of $\hat{\phi}$ is no slower than the decay rate of $\hat{f}$ for any other $f \in V \setminus \{0\}$, even though this decay rate may be significantly smaller than the approximation order of V.

The refinable functions considered in the previous paragraph are compactly supported, and their shifts are orthonormal. Our discussion, though, can be carried out under much weaker conditions on the refinable ϕ. In what follows, we assume that ϕ is univariate, that its mask m_0 (defined by $\hat{\phi} = m_0(\cdot/2)\hat{\phi}(\cdot/2)$) is continuous, that $m_0(0) = 1$, and that m_0 vanishes only on a set of measure zero. While we do not assume the shifts of ϕ to be L_2-stable, we may still invoke Theorem 2.14 of [1] to conclude that any non-zero $f \in V$ can be written as $\hat{f} = \alpha\hat{\phi}$, for some 2π-periodic measurable α. Since $f \neq 0$, we can then find a subset $E \subset [-\pi, \pi]$ of positive measure so that $|\alpha| \geqslant \delta > 0$ on E. This implies that $|\hat{f}| \geqslant \delta|\hat{\phi}|$ on $E + 2\pi\mathbb{Z}$, hence reducing the problem to studying the decay of $\sigma_E\hat{\phi}$, with $\sigma_E = \sum_{k\in\mathbb{Z}} \chi_{E+2k\pi}$ the support function of $E + 2\pi\mathbb{Z}$. The hope for a rigid connection between (s) and (ao) was based on the idea that there might exist a set E such that the decay of $\sigma_E\hat{\phi}$ is faster than of $\hat{\phi}$. Here, we define the *decay rate* of $\hat{\phi}$ as the largest λ that satisfies $|\hat{\phi}(\xi)| \leqslant C(1 + |\xi|)^{-\lambda}$.

Lower bounds on the above parameter λ can be obtained by inspecting the values of m_0 at non-trivial invariant cycles $\{\xi, \tau\xi, \ldots, \tau^{n-1}\xi\}$ (n integer > 1) of the "doubling operator"

$$\tau : \xi \mapsto 2\xi \bmod 2\pi, \tag{1}$$

as shown in [3, 9]; see also Sect 7.1.2 of [6]. More precisely, if $\tau^n\xi = \xi$ for some $\xi \in \mathbb{R}$, and if we define γ by

$$\prod_{j=0}^{n-1} |m_0(\tau^j\xi)| =: 2^{-n\gamma} > 0,$$

then $|\hat{\phi}(2^{kn+1}\xi)| \geqslant C'(2^{kn+1}|\xi| + 1)^{-\gamma}$, for some $C' > 0$. For the family of scaling functions constructed in [5], one can moreover show that the parameter γ associated with the cycle $\{2\pi/3, 4\pi/3\}$ not only provides a lower bound on λ, but actually *equals* λ.

These same invariant cycles turn out to be crucial for our question here. We have

PROPOSITION 3. *Let V be a univariate 2-refinable space generated by a function ϕ with mask m_0 : $\hat{\phi} = m_0(\cdot/2)\hat{\phi}(\cdot/2)$. Assume that m_0 is continuous, and vanishes almost nowhere. Let τ be the doubling operator from* (1), *and, for some integer n, let $0 < \xi_0 < 2\pi$ be an invariant point of τ^n : $\tau^n\xi_0 = \xi_0$, for which $m_0(\tau^j\xi_0) \neq 0$, $\forall j$. Define γ by $2^{-n\gamma} = \prod_{j=0}^{n-1} |m_0(\tau^j\xi_0)|$. Suppose also that $|\hat{\phi}| \geqslant c > 0$ around ξ_0. Then, there exists an increasing sequence of integers $(n_k)_k$, so that, for all $\epsilon > 0$, we can find an integer K, a constant C and a set S of arbitrarily small measure, such that, for all $k > K$ and all $\xi \in [-\pi, \pi] \setminus S$,*

$$|\hat{\phi}(\xi + 2\pi n_k)| \geqslant Cn_k^{-(\gamma+\epsilon)}. \tag{2}$$

Proof. 1. By the assumption made on ξ_0, $\xi_0 = 2\pi l/(2^n - 1)$, for some integer $l \in \{1, \ldots, 2^n - 1\}$. Define then the sequence of integers $n_0 = 0$, $n_k = 2^n n_{k-1} + l = l(2^{nk} - 1)/(2^n - 1)$.

2. Define $M_0(\xi) := \prod_{j=0}^{n-1} m_0(2^j\xi)$. Then M_0 is 2π-periodic, and $\hat{\phi} = M_0(\cdot/2^n)\hat{\phi}(\cdot/2^n)$. In particular,

$$\hat{\phi}(\cdot + 2\pi n_k) = M_0\left(\frac{\cdot + 2\pi l}{2^n}\right)\hat{\phi}\left(\frac{\cdot + 2\pi l}{2^n} + 2\pi n_{k-1}\right).$$

Iterating this k times, and writing the result in terms of the affine transformation $\sigma : \xi \mapsto 2^{-n}(\xi + 2\pi l)$, we have

$$\hat{\phi}(\xi + 2\pi n_k) = \prod_{j=1}^{k} M_0(\sigma^j\xi)\hat{\phi}(\sigma^k\xi).$$

3. We are assuming that $2^{-n\gamma} = |M_0(\xi_0)| \neq 0$. Now, since σ is a contraction with fixed point ξ_0, and since M_0 is continuous at ξ_0, we can find, for $\epsilon > 0$, an integer K, such that $|M_0(\sigma^k\xi)| \geqslant 2^{-n(\gamma+\epsilon)}$, for every $k > K$, and every $\xi \in [-\pi, \pi]$. Fixing ϵ and K, we can then invoke that M_0 is continuous and vanishes only on a null-set, to find a subset $S \subset [-\pi, \pi]$, of arbitrarily small measure, such that $\prod_{j=1}^{K-1} M_0(\sigma^j\xi) \geqslant C > 0$, for every $\xi \in S' := [-\pi, \pi] \setminus S$.

4. Combining the observations from 2 and 3, we conclude that, on S' and for some constant $C' > 0$,

$$|\hat{\phi}(\xi + 2\pi n_k)| \geqslant C' 2^{-n(\gamma+\epsilon)k}|\hat{\phi}(\sigma^k\xi)|, \quad \forall k. \tag{3}$$

For all sufficiently large k, $\sigma^k S'$ lies in an arbitrarily small neighborhood of ξ_0. Since $|\hat{\phi}| \geqslant c > 0$ around ξ_0, we may then, for such large k, dispense with the expression $|\hat{\phi}(\sigma^k\xi)|$ in (3) by changing the constant C', if needed. We thus obtain from (3), that, on S' and for all sufficiently large k,

$$|\hat{\phi}(\cdot + 2\pi n_k)| \geqslant C' 2^{-n(\gamma+\epsilon)k} \geqslant C'' n_k^{-(\gamma+\epsilon)}. \blacksquare$$

It then follows from our earlier discussion that no $\hat{f}, f \in V \setminus \{0\}$, can decay at a rate faster than γ, where $2^{-\gamma}$ is the geometric average of the values assumed by $|m_0|$ on an invariant cycle of τ. Therefore, if the decay rate of $\hat{\phi}$ is known to equal the parameter γ associated with some invariant cycle of τ, then this decay rate is not exceeded by the decay rate of any $\hat{f}, f \in V \setminus \{0\}$.

Now, let ϕ be the function from the family of scaling functions constructed in [5] that provides approximation order k, k integer. It is known that the decay rate of $\hat{\phi}$ is, indeed, determined by the parameter γ associated with the invariant cycle $(2\pi/3, 4\pi/3)$ (cf. [3, 9]; see also Sect. 7.1.2 in [6]). Thus, we conclude that the decay rate of any $\hat{f}, f \in V \setminus \{0\}$, cannot exceed

$$\gamma := -\frac{1}{2} \log_2 |m_0(2\pi/3) m_0(4\pi/3)|.$$

However, this last value behaves asymptotically like $k(1 - \frac{1}{2}\log_2 3)$, and hence, by selecting k sufficiently large, we obtain that the gap between the highest possible degree of smoothness in a refinable space and its approximation order can be arbitrarily large.

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REFERENCES

1. C. de Boor, R. A. DeVore, and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$, *Trans. Amer. Math. Soc.* **341** (1994), 787–806.
2. C. de Boor, K. Höllig, and S. D. Riemenschneider, "Box Splines," Springer-Verlag, Berlin/New York, 1992.
3. A. Cohen and J. P. Conze, Régularité des bases d'ondelettes et mesures ergodiques, *Rev. Mat. Iberoamericana* **8** (1992), 351–366.
4. A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, "Stationary Subdivision," Memoirs of the American Mathematical Society, Vol. 453, Providence, RI, 1991.
5. I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 909–996.
6. I. Daubechies, "Ten lectures on wavelets," CBMS Conference Series in Applied Mathematics, Vol. 61, SIAM, Philadelphia, 1992.
7. Y. Meyer, "Ondelettes et Opérateurs I: Ondelettes", Hermann, Paris, 1990.
8. A. Ron, Smooth functions in and approximation orders of refinable spaces, manuscript 1995.
9. H. Volkmer, On the regularity of wavelets, *IEEE Trans. Inf. Th.* **38** (1992), 872–876.